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Approximate roots

Approximate solutions of polynomial by using Newton's method

Let $f(x)=0$ be a polynomial equation.

Let us assume that α be a root of $f(x)=0$

Let x_0 be a number neighbourhood to α

and ~~let~~ h be a small number such that $\alpha = x_0 + h$

Since α is a root of $f(x)=0$

$$\therefore f(\alpha)=0$$

$$\Rightarrow f(x_0+h)=0 \longrightarrow (1)$$

By Taylor's Series

$$f(x_0+h) = f(x_0) + \frac{h}{1!} f'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots$$

$$0 = f(x_0) + \frac{h}{1!} f'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots \quad \{\because \text{from } (1)$$

Since h is very small,

$\therefore$ h^2, h^3 ... are negligible small quantities

$$\therefore 0 = f(x_0) + \frac{h}{1!} f'(x_0)$$

$$\Rightarrow -f(x_0) = \frac{h}{1!} f'(x_0)$$

$$\therefore \boxed{h = -\frac{f(x_0)}{f'(x_0)}} \longrightarrow (2)$$

with this h value, $x_0 + h \neq \alpha$

but $x_0 + h$ is closer to α than x_0.

Assume $x_0 + h = x_1$ (Say)

$$\Rightarrow \boxed{h = x_1 - x_0} \rightarrow (3)$$

Substitute (3) in (2)

$$(2) \Rightarrow h = -\frac{f(x_0)}{f'(x_0)}$$

$$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

Thus x_1 is a better approximation for α than x_0

Similarly $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

Here x_2 is a better approximation for α than x_1 etc..

Thus, the approximation x_n is given by

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

Example: (1) Find the real roots of $x^3 + 3x - 1 = 0$ correct to two decimal places by using Newton's method.

Solution Let $f(x) = x^3 + 3x - 1$

$f(0) = (0)^3 + 3(0) - 1 = -1$ (negative)

$f(1) = (1)^3 + 3(1) - 1 = 3$ (Positive)

$\therefore$ The given equation has a root between 0 and 1

Let $x_0 = 0$, then x_0 is an approximation for the root.

A better approximation x_1 is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\boxed{\begin{aligned} f(x) &= x^3 + 3x - 1 \\ f'(x) &= 3x^2 + 3(1) - 0 \\ f'(x) &= 3x^2 + 3 \end{aligned}}$$

$$= 0 - \frac{f(0)}{f'(0)}$$

$$= 0 - \frac{(0)^3 + 3(0) - 1}{3(0)^2 + 3}$$

$$= 0 - \frac{(-1)}{3}$$

$$\boxed{x_1 = \frac{1}{3}}$$

A better approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= \frac{1}{3} - \frac{f(1/3)}{f'(1/3)}$$

$$= \frac{1}{3} - \frac{\left(\frac{1}{3}\right)^3 + 3\left(\frac{1}{3}\right) - 1}{3\left(\frac{1}{3}\right)^2 + 3}$$

$$= \frac{1}{3} - \frac{\frac{1}{27} + 1 - 1}{3\left(\frac{1}{9}\right) + 3}$$

$$= \frac{1}{3} - \frac{\left(\frac{1}{27}\right)}{\left(\frac{1}{3} + 3\right)}$$

$$= \frac{1}{3} - \left(\frac{1/27}{10/3}\right)$$

$$= \frac{1}{3} - \left(\frac{1}{27} \times \frac{3}{10}\right)$$

$$= \frac{1}{3} - \frac{1}{90}$$

$$= \frac{30 - 1}{90}$$

$$= \frac{29}{90}$$

$$= 0.32222$$

$$\boxed{x_2 = 0.32}$$ (Correct to two decimal places)

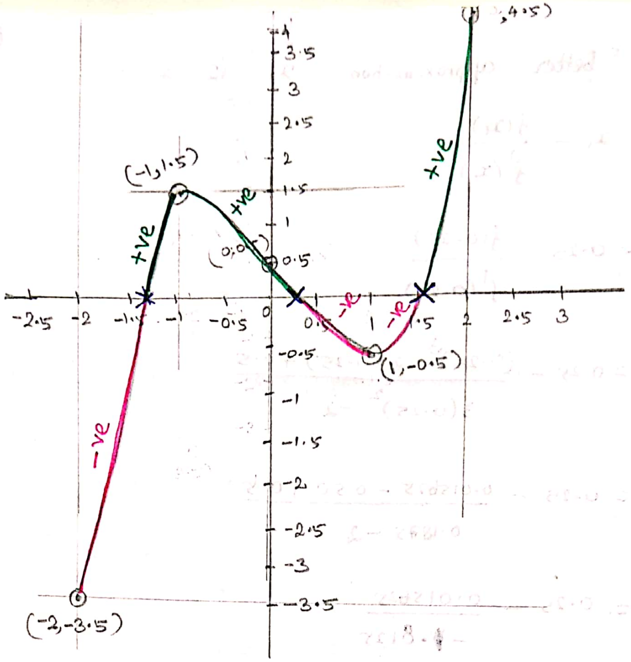
$\therefore$ 0.32 is a real root of $x^3 + 3x - 1 = 0$

Example (2) ~~Solve the~~ Find the smallest positive root of $x^3 - 2x + 0.5 = 0$ using Newton's method.

Solution

Let $f(x) = x^3 - 2x + 0.5$

x	$f(x) = x^3 - 2x + 0.5$		
-2	$f(-2) = -8 + 4 + 0.5 = -3.5$	negative ~~Positive~~	$(-2, -3.5)$ ~~Root exist~~
-1	$f(-1) = -1 + 2 + 0.5 = 1.5$	Positive	$(-1, 1.5)$ ~~No root~~
0	$f(0) = 0 - 0 + 0.5 = 0.5$	Positive	$(0, 0.5)$
1	$f(1) = 1 - 2 + 0.5 = -0.5$	negative	$(1, -0.5)$
2	$f(2) = 8 - 4 + 0.5 = 4.5$	Positive	$(2, 4.5)$



From the graph,

there is a negative root in $(-2,-1)$

· Positive root in $(0,1)$

Positive root in $(1,2)$

The smaller positive root lies between 0 and 1

Let us assume $x_0 = 0$

then x_0 is an approximation for this root.

A better approximation x_1 is given by

$$\boxed{\begin{aligned} f(x) &= x^3 - 2x + 0.5 \\ f'(x) &= 3x^2 - 2 \end{aligned}}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0 - \frac{f(0)}{f'(0)}$$

$$= 0 - \frac{0^3 - 2(0) + 0.5}{3(0)^2 - 2}$$

$$= \frac{-0.5}{-2}$$

$$\boxed{x_1 = 0.25}$$

A Still better approximation x_2 is,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.25 - \frac{f(0.25)}{f'(0.25)}$$

$$= 0.25 - \frac{(0.25)^3 - 2(0.25) + 0.5}{3(0.25)^2 - 2}$$

$$= 0.25 - \frac{0.015625 - 0.50 + 0.5}{0.1875 - 2}$$

$$= 0.25 - \frac{0.015625}{-1.8125}$$

$$= 0.25 + 0.0086$$

$$\boxed{x_2 = 0.2586}$$

$\therefore$ The required smallest positive root is 0.2586

Example (3) Find the root of $x^4 - x - 10 = 0$ which is nearer to $x = 2$ correct to three decimal places by Newton's method.

Solution:

let $f(x) = x^4 - x - 10$

$f'(x) = 4x^3 - 1$

let us assume that $x_0 = 2$

A. Better approximation is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 2 - \frac{f(2)}{f'(2)}$$

$f(x) = x^4 - x - 10$

$f(2) = 2^4 - 2 - 10$

$= 16 - 2 - 10$

$\boxed{f(2) = 4}$

$f'(x) = 4x^3 - 1$

$f'(2) = 4(2)^3 - 1$

$= 4(8) - 1$

$= 32 - 1$

$\boxed{f'(2) = 31}$

$$= 2 - \frac{4}{31}$$

$$= 2 - 0.2$$

$$\boxed{x_1 = 1.8}$$

Similarly the successive approximations are,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.8 - \frac{f(1.8)}{f'(1.8)}$$

$$= 1.8 - \frac{(1.8)^4 - 1.8 - 10}{4(1.8)^3 - 1}$$

$$= 1.8 - \frac{10.4976 - 1.8 - 10}{23.328 - 1}$$

$$= 1.8 - \frac{(-1.3024)}{22.328}$$

$$= 1.8 + 0.05833$$

$$\boxed{x_2 = 1.8583}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.8583 - \frac{f(1.8583)}{f'(1.8583)}$$

$$= 1.8583 - \frac{(1.8583)^4 - (1.8583) - 10}{4(1.8583)^3 - 1}$$

$$\boxed{x_3 = 1.8555}$$

$\therefore$ The root of x which is nearer to 2 ~~and~~ is $x_3 = 1.856$ (correct to 3 decimals)

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Example 4 : Find the positive root of the equation $x^3-2x^2-3x-4=0$ correct to two decimal places using Newton's Raphson method.

Solution

Since $f(3)=(3)^3-2(3)^2-3(3)-4$

$= 27-18-9-4$

$= 27-31$

$\boxed{f(3)=-4}$ (negative)

$f(4)=(4)^3-2(4)^2-3(4)-4$

$= 64-32-12-4$

$= 64-48$

$\boxed{f(4)=16}$ (Positive)

$\therefore$ The given equation has a positive root between 3 and 4

Let $x_0 = 3$

By Newton's Raphson method, better approximation x_1 is given by

$$\boxed{\begin{array}{l} f(x)=x^3-2x^2-3x-4 \\ f'(x)=3x^2-4x-3 \end{array}}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 3-\frac{f(3)}{f'(3)}$$

$f'(3)=3(3)^2-4(3)-3$

$=3(9)-12-3$

$=27-12-3$

$\boxed{f'(3)=12}$

$$\therefore = 3-\left(\frac{-4}{12}\right)$$

$$= 3+\frac{1}{3}$$

$$= \frac{10}{3}$$

$$\boxed{x_1 = 3.333}$$

The Successive approximations are,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 3.333 - \frac{f(3.333)}{f'(3.333)}$$

$$= 3.333 - \frac{0.758}{16.947}$$

$$= 3.333 - 0.0447$$

$$\boxed{x_2 = 3.285}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 3.285 - \frac{f(3.285)}{f'(3.285)}$$

$$= 3.285 - \frac{0.0128}{16.234}$$

$$= 3.285 - 0.00074$$

$$\boxed{x_3 = 3.28426}$$

$\therefore$ The required positive root correct to two decimal places is 3.28

Example (5) Using Newton's method, establish the formula $x_n = \frac{1}{2}\left(x_{n-1} + \frac{N}{x_{n-1}}\right)$ to find the square root of N. Hence find $\sqrt{29}$ correct to four decimal places.

Solution:

Let $x = \sqrt{N}$

Squaring on both sides

$$x^2 = N$$

$$x^2 - N = 0$$

$$\Rightarrow f(x) = x^2 - N$$

$$f'(x) = 2x$$

By Newton's method,

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

$$= x_{n-1} - \left[\frac{x_{n-1}^2 - N}{2x_{n-1}}\right]$$

{∵ By taking LCM

$$= \frac{2x_{n-1}^2 - x_{n-1}^2 + N}{2x_{n-1}}$$

$$\boxed{x_n = \frac{1}{2}\left(\frac{x_{n-1}^2 + N}{x_{n-1}}\right)}$$

$$= \frac{1}{2}\left(\frac{x_{n-1}^2}{x_{n-1}} + \frac{N}{x_{n-1}}\right)$$

$$\boxed{x_n = \frac{1}{2}\left(x_{n-1} + \frac{N}{x_{n-1}}\right)}$$

Put $N = 29$

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{29}{x_{n-1}}\right) \longrightarrow (1)$$

Since $\sqrt{29}$ is nearer to 5,

we take $x_0 = 5$ and setting $n=1$

we get,

$$(1) \Rightarrow x_1 = \frac{1}{2}\left(x_0 + \frac{29}{x_0}\right)$$

$$= \frac{1}{2}\left(5 + \frac{29}{5}\right)$$

$$= \frac{1}{2}\left(\frac{54}{5}\right)$$

$$= \frac{27}{5}$$

$$\boxed{x_1 = 5.4}$$

~~n=2, $x_2 = \frac{1}{2}x$~~

when $n=2$,

$$(1) \Rightarrow x_2 = \frac{1}{2}\left(x_1 + \frac{29}{x_1}\right)$$

$$= \frac{1}{2}\left(5.4 + \frac{29}{5.4}\right)$$

$$= \frac{1}{2}(5.4 + 5.3704)$$

$$= \frac{1}{2}(10.7704)$$

$$\boxed{x_2 = 5.3852}$$

~~$x_3 =$~~ when $n=3$,

$$(1) \Rightarrow x_3 = \frac{1}{2}\left(x_2 + \frac{29}{x_2}\right)$$

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$$x_3 = \frac{1}{2}\left(5.3852 + \frac{29}{5.3852}\right)$$

$$= \frac{1}{2}(5.3852 + 5.3851)$$

$$= \frac{1}{2}(10.7703)$$

$$\boxed{x_3 = 5.3852}$$

$\therefore$ The $\sqrt{29} = 5.3852$ (approximately)

Assignment

① Find by Newton's method the positive root of the equation $2x^3 - 3x - 6 = 0$ which lies between 1 and 2.

② Using Newton's method, obtain the formula

$$x_n = \frac{1}{3}\left(2x_{n-1} + \frac{N}{x_{n-1}^2}\right)$$ to find $N^{1/3}$.

and deduce the value for $(29)^{1/3}$ correct to three decimal places.